Polarization Relations and Dispersion Equations for Anisotropic Moving Media

Helmut Hebenstreit and Kurt Suchy
Institut für Theoretische Physik der Universität Düsseldorf

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Dedicated to Professor L. Bossy on the occasion of his 60th birthday

For media anisotropic (but not bi-anisotropic) in the comoving frame polarization relations and dispersion equations are derived using bilinear forms and quadratic forms, respectively. Specializations for media electrically anisotropic but magnetically isotropic (or vice versa) are given using (left and right) eigenvectors and eigenvalues of the material tensors.

1. Introduction

Dispersion equations for electromagnetic waves in anisotropic media are mostly represented as the vanishing of the determinant of the matrix of a Fourier transformed system of wave equations. The ratios of subdeterminants are then used to represent the corresponding polarization relations. This holds for media at rest as well as for moving media. In the latter case a four-dimensional covariant representation of the fields and of the material properties is adequate. Then the calculation of the matrix elements in covariant form is rather involved [1]. Therefore another method is used in the present paper, viz. the representation of the dispersion equation by quadratic forms and of the polarization relations by bilinear forms. This will be done in a three-dimensional form for media at rest and in a four-dimensional form for moving media. In each of the following Sects. 2 through 7 the three-dimensional calculations will be given at the beginning and are then generalized to four dimensions. The last Sect. 8 is devoted to a transformation of polarizations based on Lorentz transformation of field components.

Throughout the paper three-dimensional vectors and tensors are written in symbolic notation (with $\mathbf{I}$ as unit tensor), four-dimensional vectors and tensors in index notation with Greek indices running from 0 to 3. Dashed indices are merely labels counting vectors and eigenvalues. Quantities in the comoving frame are denoted by dashs. SI units are used, a flat space-time is assumed and the media are anisotropic but not bi-anisotropic in the comoving frame.

Reprint requests to Prof. Dr. K. Suchy, Institut für Theoretische Physik der Universität, Universitätsstr. 1, D-4000 Düsseldorf 1.

2. Maxwell's Equations for Four-dimensional Fields

In this section we shall generalize the derivation of the three-dimensional algebraic Maxwell equations

$$\boldsymbol{k} \times \boldsymbol{\mu}^{-1} \cdot \boldsymbol{B} + \omega (\boldsymbol{\epsilon} + i\, \boldsymbol{\sigma}/\omega) \cdot \boldsymbol{E} = 0\,, \qquad (2.1\,\mathrm{a})$$

$$\boldsymbol{k} \times \boldsymbol{E} - \omega\, \boldsymbol{B} = 0 \qquad (2.1\,\mathrm{b})$$

from the original Maxwell equations

$$\frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{H} - \frac{\partial}{\partial t} \boldsymbol{D} = \boldsymbol{j}\,, \quad \frac{\partial}{\partial \boldsymbol{x}} \times \boldsymbol{E} + \frac{\partial}{\partial t} \boldsymbol{B} = 0\,, \quad (2.2)$$

and the constitutive relations

$$\boldsymbol{B} = \boldsymbol{\mu} \cdot \boldsymbol{H}, \quad \boldsymbol{D} = \boldsymbol{\epsilon} \cdot \boldsymbol{E}, \quad \boldsymbol{j} = \boldsymbol{\sigma} \cdot \boldsymbol{E}\,, \qquad (2.3)$$

using the plane wave ansatz $\exp i(\boldsymbol{k} \cdot \boldsymbol{x} - \omega\, t)$. In (2.1a) the tensors $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ appear only in the combination

$$\check{\boldsymbol{\epsilon}} := \boldsymbol{\epsilon} + i\, \boldsymbol{\sigma}/\omega\,. \qquad (2.4)$$

To write Maxwell's Eqs. (2.2) in covariant form we introduce the excitation tensor G and the field tensor F defined by

$$G^{\nu\gamma} := \left[\begin{array}{c:c} 0 & c\boldsymbol{D} \\ \hdashline -\,c\boldsymbol{D} & \mathbf{I} \times \boldsymbol{H} \end{array}\right], \quad F_{\lambda\mu} := \left[\begin{array}{c:c} 0 & -\boldsymbol{E} \\ \hdashline \boldsymbol{E} & \mathbf{I} \times c\boldsymbol{B} \end{array}\right]. \qquad (2.5)$$

Together with the four-dimensional current density $j^\nu := [c\varrho \,\vdots\, \boldsymbol{j}]$ and the tensor ([2], Chapter 11.9)

$$\mathscr{F}^{\nu\gamma} := \tfrac{1}{2}\, \varepsilon^{\nu\gamma\lambda\mu} F_{\lambda\mu} \qquad (2.6)$$

dual to $F_{\lambda\mu}$ (with $\varepsilon^{\nu\gamma\lambda\mu}$ as the Levi-Civita symbol) Maxwell's equations (2.2) in covariant form are (for a flat space-time)

$$\partial_\gamma G^{\nu\gamma} = j^\nu, \quad \partial_\gamma \mathscr{F}^{\nu\gamma} = 0\,. \qquad (2.7)$$

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The covariant forms of the constitutive relations (2.3) are ([3], Chapter 4; [4], eq. 2.1)

$$G^{\nu\gamma} = \tfrac{1}{2} c\, \lambda^{\nu\gamma}{}_{\lambda\mu} F^{\lambda\mu} , \tag{2.8a}$$

$$(\delta^{\nu}_{\mu} + u^{\nu} u_{\mu})\, j^{\mu} = \tfrac{1}{2} \sigma^{\nu}{}_{\lambda\mu} F^{\lambda\mu} \tag{2.8b}$$

with $\boldsymbol{v}$ as velocity and

$$u^{\nu} := \frac{\gamma}{c} [c \,\vdots\, \boldsymbol{v}] =: (1 - v^2/c^2)^{-1/2} [1 \,\vdots\, \boldsymbol{v}/c] \tag{2.8c}$$

as the dimensionless four-velocity ($u^{\nu} u_{\nu} = -1$).

The material tensors λ and σ of order four and three, respectively, can be represented by four-dimensional material tensors ε, $\varkappa$, σ of order two, which are closely related to the three-dimensional dielectric, permeability and conductivity tensors $\boldsymbol{\epsilon}$, $\boldsymbol{\mu}^{-1}$, $\boldsymbol{\sigma}$, respectively. In the case of a medium anisotropic (but not bi-anisotropic) in the comoving frame the constitutive tensors λ and σ are given through ([3], Chapter 4; [4], Eq. (2.7))

$$\begin{aligned} \lambda^{\nu\gamma}{}_{\lambda\mu} = {} & (\varepsilon^{\gamma}{}_{\lambda} u_{\mu} - \varepsilon^{\gamma}{}_{\mu} u_{\lambda}) u^{\nu} \\ & - (\varepsilon^{\nu}{}_{\lambda} u_{\mu} - \varepsilon^{\nu}{}_{\mu} u_{\lambda}) u^{\gamma} \\ & + \varepsilon_0 \mu_0 \varkappa_{\sigma}{}^{\varrho} \varepsilon^{\nu\gamma\varphi\sigma} \varepsilon_{\lambda\mu\psi\varrho} u_{\varphi} u^{\psi} , \end{aligned} \tag{2.9a}$$

$$\sigma^{\nu}{}_{\lambda\mu} = \sigma^{\nu}{}_{\lambda} u_{\mu} - \sigma^{\nu}{}_{\mu} u_{\lambda} . \tag{2.9b}$$

In the comoving frame (denoted by a dash) the four-dimensional material tensors ε', $\varkappa'$ and σ' of order two are related to the three-dimensional dielectric, permeability and conductivity tensors $\boldsymbol{\epsilon}'$, $\boldsymbol{\mu}'$ and $\boldsymbol{\sigma}'$ by ([4], Eqs. (2.9), (2.11), (4.12), (4.13))

$$\varepsilon'^{\gamma}{}_{\lambda} := \left[\begin{array}{c|c} \bar{\varepsilon} & 0 \\ \hline 0 & \boldsymbol{\epsilon}' \end{array}\right], \quad \varkappa'_{\sigma}{}^{\varrho} := \left[\begin{array}{c|c} \overline{\mu^{-1}} & 0 \\ \hline 0 & \boldsymbol{\mu}'^{-1} \end{array}\right], \tag{2.10a, b}$$

$$\sigma'^{\nu}{}_{\lambda} := \left[\begin{array}{c|c} \bar{\sigma} & 0 \\ \hline 0 & \boldsymbol{\sigma}' \end{array}\right] \tag{2.10c}$$

with arbitrary values for the time components $\bar{\varepsilon}$, $\overline{\mu^{-1}}$, $\bar{\sigma}$.

The constitutive relation (2.8b) merely connects the conducting part $(\delta^{\nu}_{\mu} + u^{\nu} u_{\mu})\, j^{\mu}$ of the current density j^{μ} with the electromagnetic field tensor $F^{\lambda\mu}$. An equivalent constitutive relation for the total current density j^{μ}, i.e. the sum of the conducting and the convective part $-u^{\nu} u_{\mu} j^{\mu}$, would lead to more complicated expressions ([4], Sect. 3). This suggests to project Maxwell's equations (2.7) into subspaces perpendicular and parallel to the (dimensionless) four-velocity u^{ν}, i.e.

$$(\delta^{\nu}_{\mu} + u^{\nu} u_{\mu})\, \partial_{\gamma} G^{\mu\gamma} = (\delta^{\nu}_{\mu} + u^{\nu} u_{\mu})\, j^{\mu} , \tag{2.11a}$$

$$u_{\mu}\, \partial_{\gamma} G^{\mu\gamma} = u_{\mu} j^{\mu} , \tag{2.11b}$$

$$(\delta^{\nu}_{\mu} + u^{\nu} u_{\mu})\, \partial_{\gamma} \mathscr{F}^{\mu\gamma} = 0 , \tag{2.11c}$$

$$u_{\mu}\, \partial_{\gamma} \mathscr{F}^{\mu\gamma} = 0 . \tag{2.11d}$$

In the comoving frame Eqs. (2.11a and c) are the usual three-dimensional Maxwell curl-equations (2.2), the remaining ones (2.11b and d) are the div-equations.

With the four-dimensional propagation vector $k_{\alpha} := [-\omega/c \,\vdots\, \boldsymbol{k}]$ the plane wave ansatz reads $\exp(i k_{\alpha} x^{\alpha})$. It replaces the differential operator ∂_{γ} in Maxwell's equations (2.11) by $i k_{\gamma}$. To express the results of the combination of Maxwell's equations (2.11) with the constitutive relations (2.8) we first represent the convection current density $u_{\mu} j^{\mu}$ (using the algebraic continuity equation $k_{\nu} j^{\nu} = 0$ and the constitutive relation (2.8b)) in the form

$$\begin{aligned} u_{\mu} j^{\mu} & = \frac{1}{k_{\gamma} u^{\gamma}} k_{\nu} (\delta^{\nu}_{\mu} + u^{\nu} u_{\mu})\, j^{\mu} \\ & = \frac{1}{k_{\gamma} u^{\gamma}} k_{\nu} \frac{1}{2} \sigma^{\nu}{}_{\lambda\mu} F^{\lambda\mu} . \end{aligned} \tag{2.12}$$

Analogous to $\breve{\boldsymbol{\epsilon}} := \boldsymbol{\epsilon} + i\, \boldsymbol{\sigma}/\omega$ (2.4) we introduce the four-dimensional tensor $\breve{\varepsilon}$ by

$$\breve{\varepsilon}^{\nu}{}_{\lambda} := \varepsilon^{\nu}{}_{\lambda} - \frac{i}{c\, k_{\gamma} u^{\gamma}} \sigma^{\nu}{}_{\lambda} . \tag{2.13}$$

Using furthermore the definition (2.6) of the dual field tensor $\mathscr{F}$ we obtain finally the following set of algebraic equations for the field tensor F:

$$\begin{aligned} [& \varepsilon^{\nu\gamma\varphi\sigma} k_{\gamma} u_{\varphi} \varepsilon_0 \mu_0 \varkappa_{\sigma}{}^{\varrho} u^{\psi} \varepsilon_{\lambda\mu\psi\varrho} \\ & - k_{\gamma} u^{\gamma} (\breve{\varepsilon}^{\nu}{}_{\lambda} u_{\mu} - \breve{\varepsilon}^{\nu}{}_{\mu} u_{\lambda})]\, F^{\lambda\mu} = 0 , \end{aligned} \tag{2.14a}$$

$$k_{\gamma} \breve{\varepsilon}^{\gamma}{}_{\lambda} u_{\mu} F^{\lambda\mu} = 0 , \tag{2.14b}$$

$$(\delta^{\nu}_{\beta} + u^{\nu} u_{\beta})\, \varepsilon^{\beta\gamma\lambda\mu} k_{\gamma} F_{\lambda\mu} = 0 , \tag{2.14c}$$

$$u_{\beta}\, \varepsilon^{\beta\gamma\lambda\mu} k_{\gamma} F_{\lambda\mu} = 0 . \tag{2.14d}$$

The last step in this section is the representation of the field tensor F in (2.14) by two four-fields

$$E^{\lambda} := F^{\lambda\mu} u_{\mu} = \gamma \left[\frac{\boldsymbol{v}}{c} \cdot \boldsymbol{E} \,\vdots\, \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \right], \tag{2.15a}$$

$$\begin{aligned} c\, B^{\lambda} & := \mathscr{F}^{\lambda\mu} u_{\mu} = \tfrac{1}{2} \varepsilon^{\lambda\mu\varphi\psi} u_{\mu} F_{\varphi\psi} \\ & = \gamma \left[\boldsymbol{v} \cdot \boldsymbol{B} \,\vdots\, c\, \boldsymbol{B} - \frac{\boldsymbol{v}}{c} \times \boldsymbol{E} \right] \end{aligned} \tag{2.15b}$$

in the following manner:

$$F_{\lambda\mu} = u_\lambda E_\mu - E_\lambda u_\mu - \varepsilon_{\lambda\mu\varphi\psi} u^\varphi c B^\psi . \quad (2.15\text{c})$$

Insertion of this representation into Eqs. (2.14) yields the algebraic Maxwell equations for the four-fields:

$$\varepsilon^{\nu\gamma\varphi\sigma} k_\gamma u_\varphi \varepsilon_0 \mu_0 \varkappa_\sigma{}^\varrho c B_\varrho + k_\gamma u^\gamma \check{\varepsilon}^\nu{}_\lambda E^\lambda = 0 , \quad (2.16\text{a})$$

$$k_\gamma \check{\varepsilon}^\gamma{}_\lambda E^\lambda = 0 , \quad (2.16\text{b})$$

$$\varepsilon^{\nu\gamma\lambda\mu} k_\gamma u_\lambda E_\mu - k_\gamma u^\gamma c B^\nu = 0 , \quad (2.16\text{c})$$

$$k_\gamma B^\gamma = 0 . \quad (2.16\text{d})$$

In the comoving frame equations (2.16) are the three-dimensional algebraic Maxwell equations. The spatial components of (2.16a, c) are Maxwell's algebraic curl-equations (2.1) written with the tensor $\check{\boldsymbol{\varepsilon}} := \boldsymbol{\varepsilon} + i\,\boldsymbol{\sigma}/\omega$ (2.4). The time components $\nu = 0$ of Eqs. (2.16a, c) yield the identity $0 = 0$. Equation (2.16b) expresses the transversality of the effective displacement vector $\check{\boldsymbol{\varepsilon}}' \cdot \boldsymbol{E}'$ in the comoving frame, i.e.

$$\boldsymbol{k}' \cdot \check{\boldsymbol{\varepsilon}}' \cdot \boldsymbol{E}' = 0 ,$$

while Eq. (2.16d) becomes

$$\boldsymbol{k}' \cdot \boldsymbol{B}' = 0 ,$$

i.e. the transversality condition for $\boldsymbol{B}'$ in the comoving frame.

3. Algebraic Wave Equations for Electric or Magnetic Field Components

To establish a dispersion equation and polarization relations from the three-dimensional Maxwell equations (2.1) one must eliminate either $\boldsymbol{B}$ or $\boldsymbol{E}$. The choice depends on the material properties. This will be discussed in Section 4. Upon eliminating $\boldsymbol{B} = (\boldsymbol{k}/\omega) \times \boldsymbol{E}$ (2.1b) one obtains

$$\left[\frac{\check{\boldsymbol{\varepsilon}}}{\varepsilon_0} + \boldsymbol{n} \times \mu_0 \boldsymbol{\varkappa} \times \boldsymbol{n}\right] \cdot \boldsymbol{E} = 0 \quad (3.1\text{a})$$

with the refractive index vector

$$\boldsymbol{n} := \frac{c}{\omega} \boldsymbol{k} = \frac{\boldsymbol{k}}{\omega\sqrt{\varepsilon_0 \mu_0}} \quad \text{and} \quad \boldsymbol{\varkappa} := \boldsymbol{\mu}^{-1} . \quad (3.1\text{b})$$

The decomposition of the reciprocal permittivity tensor

$$\boldsymbol{\varkappa} = \varkappa \mathbf{I} + (\boldsymbol{\varkappa} - \varkappa \mathbf{I}) =: \varkappa[\mathbf{I} + (\hat{\boldsymbol{\varkappa}} - \mathbf{I})] \quad (3.2)$$

into an isotropic part $\varkappa \mathbf{I}$ and an anisotropic part $\varkappa(\hat{\boldsymbol{\varkappa}} - \mathbf{I})$ allows to introduce the longitudinal projector $\boldsymbol{n}\boldsymbol{n}/n^2 = \mathbf{I} + \boldsymbol{n} \times \mathbf{I} \times \boldsymbol{n}/n^2$ needed in Section 4:

$$[\hat{\boldsymbol{\varepsilon}} + \boldsymbol{n} \times (\hat{\boldsymbol{\varkappa}} - \mathbf{I}) \times \boldsymbol{n} - n^2 \mathbf{I} + \boldsymbol{n}\,\boldsymbol{n}] \cdot \boldsymbol{E} = 0 \quad (3.3\text{a})$$

with the relative effective dielectric tensor

$$\hat{\boldsymbol{\varepsilon}} := \check{\boldsymbol{\varepsilon}}/\varepsilon_0 \mu_0 \varkappa . \quad (3.3\text{b})$$

On the other hand the elimination of

$$\boldsymbol{E} = -\check{\boldsymbol{\varepsilon}}^{-1} \cdot (\boldsymbol{k}/\omega) \times \boldsymbol{H} \quad [2.1\text{a}]$$

leads to

$$\left[\frac{\boldsymbol{\mu}}{\mu_0} + \boldsymbol{n} \times \varepsilon_0 \boldsymbol{\eta} \times \boldsymbol{n}\right] \cdot \boldsymbol{H} = 0 \quad (3.4\text{a})$$

with

$$\boldsymbol{\eta} := \check{\boldsymbol{\varepsilon}}^{-1} . \quad (3.4\text{b})$$

Its decomposition

$$\boldsymbol{\eta} = \eta \mathbf{I} + (\boldsymbol{\eta} - \eta \mathbf{I}) =: \eta[\mathbf{I} + (\hat{\boldsymbol{\eta}} - \mathbf{I})] \quad (3.5)$$

allows to introduce $\boldsymbol{n}\boldsymbol{n} \cdot \boldsymbol{H}$ as follows:

$$[\hat{\boldsymbol{\mu}} + \boldsymbol{n} \times (\hat{\boldsymbol{\eta}} - \mathbf{I}) \times \boldsymbol{n} - n^2 \mathbf{I} + \boldsymbol{n}\,\boldsymbol{n}] \cdot \boldsymbol{H} = 0 \quad (3.6\text{a})$$

with the relative effective permeability tensor

$$\hat{\boldsymbol{\mu}} := \boldsymbol{\mu}/\mu_0 \varepsilon_0 \eta . \quad (3.6\text{b})$$

Both algebraic wave equations (3.3a) and (3.6a) are completely equivalent. For magnetically isotropic media (with $\hat{\boldsymbol{\varkappa}} = \mathbf{I}$) Eq. (3.3a) is more convenient than (3.6a), which in turn is more advantageous than (3.3a) for electrically isotropic media (with $\hat{\boldsymbol{\eta}} = \mathbf{I}$).

The generalization of the preceding procedure for moving media means the elimination of either B^ν or E_λ from the algebraic Maxwell equations (2.16). Elimination of

$$c B_\varrho = (1/k_\gamma u^\gamma)\, \varepsilon_{\varrho\beta\mu\lambda} k^\beta u^\mu E^\lambda \quad [2.16\text{c}]$$

yields as equation for the four-field E^λ

$$\left[\frac{\check{\varepsilon}^\nu{}_\lambda}{\varepsilon_0} + \frac{k_\gamma u_\varphi}{k_\psi u^\psi} \varepsilon^{\nu\gamma\varphi\sigma} \mu_0 \varkappa_\sigma{}^\varrho \varepsilon_{\varrho\beta\mu\lambda} \frac{k^\beta u^\mu}{k_\psi u^\psi}\right] E^\lambda = 0 . \quad (3.7)$$

This is the generalization of Equation (3.1a). To introduce a projector which is the generalization of $\boldsymbol{n}\,\boldsymbol{n}/n^2$ we first decompose $\varkappa_\sigma{}^\varrho$ (2.10b) as

$$\varkappa_\sigma{}^\varrho = \varkappa\, \delta_\sigma^\varrho + (\varkappa_\sigma{}^\varrho - \varkappa\, \delta_\sigma^\varrho) =: \varkappa[\delta_\sigma^\varrho + \hat{\varkappa}_\sigma{}^\varrho - \delta_\sigma^\varrho)] \quad (3.8)$$

analogous to (3.2). But the generalization of the projector $\boldsymbol{k}\boldsymbol{k}/k^2$ is not $k^\nu k_\lambda / k_\mu k^\mu$. Instead of the four-dimensional propagation vector $k^\lambda := [\omega/c \,\vdots\, \boldsymbol{k}]$ its projection perpendicular to the four-velocity u^ν has to be used, expressed by the dimensionless four-vector

$$n^\nu := \frac{1}{-k_\gamma u^\gamma} (\delta^\nu_\lambda + u^\nu u_\lambda) k^\lambda = \frac{k^\nu}{-k_\gamma u^\gamma} - u^\nu$$
$$= \frac{\gamma}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} \left[\boldsymbol{v} \cdot \left(\boldsymbol{k} - \frac{\omega\,\boldsymbol{v}}{c^2}\right) \,\vdots\, c\left(\boldsymbol{k} - \frac{\omega\,\boldsymbol{v}}{c^2}\right) - \frac{v^2}{c}\left(\mathbf{I} - \frac{\boldsymbol{v}\,\boldsymbol{v}}{v^2}\right)\cdot\boldsymbol{k} \right]. \qquad (3.9\text{a})$$

The normalization with $-k_\gamma u^\gamma = \gamma(\omega - \boldsymbol{k}\cdot\boldsymbol{v})/c$ has been chosen to obtain in the comoving frame

$$n'^\nu = \left[0 \,\vdots\, \frac{c}{\omega'}\,\boldsymbol{k}'\right] = [0 \,\vdots\, \boldsymbol{n}'] \,, \qquad (3.9\text{b})$$

i.e. the spatial components of n'^ν are given by the refractive index vector $\boldsymbol{n}' := c\boldsymbol{k}'/\omega'$ (3.1 b) in the comoving frame.

From the definition (3.9a) of n^ν there follows its orthogonality with respect to u_ν, viz.

$$n^\nu u_\nu = 0 \qquad (3.10\text{a})$$

and the relation

$$n^\nu n_\nu = k^\nu k_\nu / (k_\gamma u^\gamma)^2 + 1 \qquad (3.10\text{b})$$

between the squared norms of n^ν and k^ν. This allows us to write

$$-k_\lambda E^\lambda = -(k_\lambda + k_\gamma u^\gamma u_\lambda) E^\lambda$$
$$= n_\lambda E^\lambda k_\gamma u^\gamma \,, \qquad (3.11)$$

where the orthogonality $u_\lambda E^\lambda = 0$ has been used, following from the definition (2.15a) of E^λ.

With this result and with the decomposition (3.8) of the tensor $\varkappa_\sigma{}^\varrho$, the introduction of the vector n^ν (3.9a), the relative effective dielectric tensor

$$\hat{\varepsilon}^\nu{}_\lambda := \frac{\check{\varepsilon}^\nu{}_\lambda}{\varepsilon_0 \mu_0 \varkappa} = \frac{1}{\varepsilon_0 \mu_0 \varkappa}\left(\varepsilon^\nu{}_\lambda - \frac{i}{c\,k_\gamma u^\gamma}\,\sigma^\nu{}_\lambda\right), \qquad (3.12)$$

and a relation ([3], Eq. (I.13.29)) for the contraction $\varepsilon^{\nu\gamma\varphi\sigma}\varepsilon_{\sigma\beta\mu\lambda}$ of two Levi-Civita tensors, Eq. (3.7) for E^λ can be written as

$$[\hat{\varepsilon}^\nu{}_\lambda + n_\gamma u_\varphi \varepsilon^{\nu\gamma\varphi\varrho} (\hat{\varkappa}_\sigma{}^\varrho - \delta^\varrho_\sigma)\,\varepsilon_{\varrho\beta\mu\lambda} n^\beta u^\mu$$
$$- n_\gamma n^\gamma \delta^\nu_\lambda + n^\nu n_\lambda] E^\lambda = 0 \,. \qquad (3.13)$$

This is the generalization of (3.3a).

The generalization of (3.6a) can be obtained in an analogous manner eliminating E^λ instead of B_ϱ and decomposing the reciprocal of $\check{\varepsilon}^\nu{}_\lambda$ instead of $\varkappa_\sigma{}^\varrho$ (3.8).

In the comoving frame the time component $\nu = 0$ of (3.13) expresses the identity $0 = 0$. The space components $\nu = 1, 2, 3$ are the three-dimensional system (3.3a) for $\boldsymbol{E}'$.

4. Polarization Relations and Dispersion Equations

To derive polarization relations and a dispersion equation from the algebraic wave equation (3.3a) without using the determinant and subdeterminants we first separate the longitudinal vector $\boldsymbol{n}\boldsymbol{n}\cdot\boldsymbol{E}$ from the rest of Equation (3.3a):

$$\mathbf{C}\cdot\boldsymbol{E} = \boldsymbol{n}\boldsymbol{n}\cdot\boldsymbol{E} \qquad (4.1\text{a})$$

with

$$\mathbf{C} := n^2\,\mathbf{I} - \hat{\boldsymbol{\varepsilon}} - \boldsymbol{n}\times(\hat{\boldsymbol{\varkappa}} - \mathbf{I})\times\boldsymbol{n} \,. \qquad (4.1\text{b})$$

Multiplication of (4.1a) with $\mathbf{C}^{-1}$ yields

$$\boldsymbol{E} = \mathbf{C}^{-1}\cdot\boldsymbol{n}\boldsymbol{n}\cdot\boldsymbol{E} \,. \qquad (4.2)$$

Because of the common factor $\boldsymbol{n}\cdot\boldsymbol{E}$ the ratios of the three components of $\boldsymbol{E}$ are given by the ratios of the three components of the vector $\mathbf{C}^{-1}\cdot\boldsymbol{n}$. The representation (4.2) of $\boldsymbol{E}$ is already the expression for the polarization relations.

At the first glance the $\boldsymbol{E}$-representation (4.2) seems to hold only for non-singular tensors $\mathbf{C}$ (4.1 b). The tensor $\mathbf{C}$ in (4.1a) becomes singular for vanishing longitudinal component $\boldsymbol{E}\cdot\boldsymbol{n}/n$ of $\boldsymbol{E}$, because then (4.1a) becomes a homogeneous system for the determination of the $\boldsymbol{E}$-components. But even in this case the adjoint of $\mathbf{C}$ does exist, defined by

$$\mathbf{C}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{C} = \mathbf{I}\det\mathbf{C} \quad \text{or}$$
$$\mathbf{A} := \mathbf{C}^{-1}\det\mathbf{C} \,. \qquad (4.3)$$

Therefore we can write the polarization relation (4.2) as

$$\boldsymbol{E} = \mathbf{A}\cdot\boldsymbol{n}\,(\boldsymbol{n}\cdot\boldsymbol{E}/\det\mathbf{C}) \,. \qquad (4.4)$$

Now the ratios of the $\boldsymbol{E}$-components are expressed by the ratios of the components of $\mathbf{A}\cdot\boldsymbol{n}$, which always exist. Their explicit calculation will be done in Section 7.

Multiplication of (4.2) with $\boldsymbol{n}$ and division by $\boldsymbol{n}\cdot\boldsymbol{E}$ yields as dispersion equation

$$\boldsymbol{n}\cdot\mathbf{C}^{-1}\cdot\boldsymbol{n} = 1 \,. \qquad (4.5)$$

According to the discussion following Eq. (4.2) the tensor $\mathbf{C}$ becomes singular for purely transverse $\boldsymbol{E}$ (4.1a) and consequently $\mathbf{C}^{-1}$ does not exist.

Multiplication of (4.5) with det $\mathbf{C}$ (or multiplication of (4.2) with $\boldsymbol{n}$ det $\mathbf{C}$) yields the more general dispersion equation

$$\boldsymbol{n} \cdot \mathbf{A} \cdot \boldsymbol{n} - \det \mathbf{C} = 0\,. \tag{4.6}$$

In an isotropic medium with $\hat{\boldsymbol{\epsilon}} = \hat{\varepsilon}\,\mathbf{I}$ and $\hat{\boldsymbol{\varkappa}} = \mathbf{I}$ we obtain $\mathbf{C} = (n^2 - \hat{\varepsilon})\,\mathbf{I}$ from the definition (4.1 b), yielding det $\mathbf{C} = (n^2 - \hat{\varepsilon})^3$ and $\mathbf{A} = (n^2 - \hat{\varepsilon})^2\,\mathbf{I}$. With det $\mathbf{C} = O(\boldsymbol{n} \cdot \boldsymbol{E})$ (4.1 a) the polarization relation (4.4) yields

$$\boldsymbol{E} \sim (n^2 - \hat{\varepsilon})^2\,\boldsymbol{n}\,, \tag{4.7}$$

and the dispersion equation (4.6) becomes

$$n^2 (n^2 - \hat{\varepsilon})^2 - (n^2 - \hat{\varepsilon})^3 = \hat{\varepsilon}\,(n^2 - \hat{\varepsilon})^2 = 0 \tag{4.8}$$

with the double solution $n^2 = \hat{\varepsilon}$. With this solution the polarization expression (4.7) yields the vanishing of the longitudinal component of $\boldsymbol{E}$.

In an isotropic medium the dispersion equation (4.8) could be expressed in the form $\hat{\boldsymbol{\epsilon}} \cdot \mathbf{A} = 0$. This product of $\hat{\boldsymbol{\epsilon}}$ and $\mathbf{A}$ suggests another form of the general dispersion equation (4.5) or (4.6) for anisotropic media. We multiply the polarization relation (4.4) with $\hat{\boldsymbol{\epsilon}}$ and obtain

$$\hat{\boldsymbol{\epsilon}} \cdot \boldsymbol{E} = \hat{\boldsymbol{\epsilon}} \cdot \mathbf{A} \cdot \boldsymbol{n}\,(\boldsymbol{n} \cdot \boldsymbol{E}/\det \mathbf{C})\,. \tag{4.9}$$

Because of the algebraic Maxwell equation (2.1 a) the vector

$$\hat{\boldsymbol{\epsilon}} \cdot \boldsymbol{E} = (\check{\boldsymbol{\epsilon}}/\varepsilon_0\,\mu_0\,\varkappa) \cdot \boldsymbol{E} = (\boldsymbol{\epsilon} + i\,\boldsymbol{\sigma}/\omega) \cdot \boldsymbol{E}/\varepsilon_0\,\mu_0\,\varkappa$$

is purely transverse. Its scalar product with $\boldsymbol{n}$ must therefore vanish and we obtain the dispersion relation in the form

$$\boldsymbol{n} \cdot \hat{\boldsymbol{\epsilon}} \cdot \mathbf{A} \cdot \boldsymbol{n} = 0\,. \tag{4.10}$$

For crystals with $\boldsymbol{\mu} = \mu_0\,\mathbf{I}$, $\boldsymbol{\sigma} = 0$, i.e. $\hat{\boldsymbol{\epsilon}} = \boldsymbol{\epsilon}/\varepsilon_0$, this form of the dispersion relation has been derived already by Gibbs ([5], p. 397, eq. 14). For isotropic media the result $n^2\,\hat{\varepsilon}\,(n^2 - \hat{\varepsilon})^2 = 0$ is equivalent to (4.8).

Instead of starting with the algebraic wave equation (3.3 a) one can as well choose its counterpart (3.6 a) where $\boldsymbol{E}$ has been eliminated instead of $\boldsymbol{H}$. Then $\boldsymbol{E}$ has to be replaced by $\boldsymbol{H}$ and in the expression (4.1 b) for the tensor $\mathbf{C}$ the tensor $\hat{\boldsymbol{\epsilon}}$ (3.3 b) and $\hat{\boldsymbol{\varkappa}}$ (3.2) have to be replaced by $\hat{\boldsymbol{\mu}}$ (3.6 b) and $\hat{\boldsymbol{\eta}}$ (3.5), respectively. If the medium is magnetically isotropic $\hat{\boldsymbol{\varkappa}} - \mathbf{I}$ vanishes and $\mathbf{C}$ (4.1 b) simplifies to $n^2\,\mathbf{I} - \hat{\boldsymbol{\epsilon}}$. If the medium has an isotropic dielectric and/or conductivity tensor, $\hat{\boldsymbol{\eta}} - \mathbf{I}$ vanishes and the corresponding tensor $\mathbf{C}$ simplifies to $n^2\,\mathbf{I} - \hat{\boldsymbol{\mu}}$. For a plasma the magnetic and dielectric properties can mostly be ignored and therefore in this case it is convenient to work with $\mathbf{C} = n^2\,\mathbf{I} - \hat{\boldsymbol{\epsilon}}$. Then $\hat{\boldsymbol{\epsilon}}$ and $\mathbf{C}$ have the same eigenvectors; this facilitates the inversion of $\mathbf{C}$, as will be shown in Section 5. In the most general case $\mathbf{C}$ (4.1 b) and the material tensors $\hat{\boldsymbol{\epsilon}}$ and $\boldsymbol{\mu}$ have no common eigenvectors (even if $\hat{\boldsymbol{\epsilon}}$ and $\boldsymbol{\mu}$ do have them). Then the inversion of $\mathbf{C}$ is rather difficult and it might be more convenient to establish the dispersion equation, as usual, by the vanishing of a determinant than by a quadratic form (4.6) or (4.10).

To generalize the derivation of the polarization relations (4.4) and the dispersion equations (4.6) and (4.10) to moving media we define a four-tensor C as the analogue to the tensor $\mathbf{C}$ (4.1 b) by means of (3.13) as

$$C^{\nu}{}_{\lambda} := n_{\gamma}\,n^{\gamma}\,\delta^{\nu}_{\lambda} - \hat{\varepsilon}^{\nu}{}_{\lambda} - n_{\gamma}\,u_{\varphi}\,\varepsilon^{\nu\gamma\varphi\sigma}\,(\hat{\varkappa}_{\sigma}{}^{\varrho} - \delta^{\varrho}_{\sigma})\,\varepsilon_{\varrho\beta\mu\lambda}\,n^{\beta}\,u^{\mu} \tag{4.11 a}$$

and its adjoint A by

$$C^{\mu}{}_{\nu}\,A^{\nu}{}_{\lambda} = A^{\mu}{}_{\nu}\,C^{\nu}{}_{\lambda} = \delta^{\mu}_{\lambda}\,\det C\,. \tag{4.11 b}$$

We write the algebraic wave equation (3.13) as

$$C^{\nu}{}_{\lambda}\,E^{\lambda} = n^{\nu}\,n_{\lambda}\,E^{\lambda} \tag{4.12}$$

analogous to (4.1 a) and contract it with $A^{\mu}{}_{\nu}$. We obtain the polarization relation

$$E^{\mu} = A^{\mu}{}_{\nu}\,n^{\nu}\,(n_{\lambda}\,E^{\lambda}/\det C) \tag{4.13}$$

as the generalization of (4.4).

Contraction with n_{μ} and multiplication with $\det C/n_{\lambda}E^{\lambda}$ yields the generalization of the dispersion equation (4.6) as

$$n_{\mu}\,A^{\mu}{}_{\nu}\,n^{\nu} - \det C = 0\,. \tag{4.14}$$

To generalize the form (4.10) of the dispersion equation we contract the polarization relation (4.13) with the tensor $\hat{\varepsilon}^{\lambda}{}_{\mu}$ and obtain

$$\hat{\varepsilon}^{\lambda}{}_{\mu}\,E^{\mu} = \hat{\varepsilon}^{\lambda}{}_{\mu}\,A^{\mu}{}_{\nu}\,n^{\nu}\,(n_{\psi}\,E^{\psi}/\det C)\,. \tag{4.15}$$

According to the algebraic Maxwell equation (2.16 a) the vector $\hat{\varepsilon}^{\nu}{}_{\lambda}E^{\lambda} = (\check{\varepsilon}^{\nu}{}_{\lambda}/\varepsilon_0\,\mu_0\,\varkappa)\,E^{\lambda}$ is perpendicular to n_{ν}, since $n_{\nu}\,\varepsilon^{\nu\gamma\varphi\sigma}\,k_{\gamma}\,u_{\varphi}$ vanishes because of the definition (3.9 a) of n_{ν}. Since the contraction of $\hat{\varepsilon}^{\lambda}{}_{\mu}E^{\mu}$ with n_{λ} vanishes, so does the right-hand side of (4.15) and we obtain the dispersion relation in the form analogous to (4.10) as

$$n_{\lambda}\,\hat{\varepsilon}^{\lambda}{}_{\mu}\,A^{\mu}{}_{\nu}\,n^{\nu} = 0\,. \tag{4.16}$$

In the comoving frame the tensor A' is expressed explicitly by

$$A'^{\mu}{}_{\nu} = \left[\begin{array}{c|c} \bar{A} & 0 \\ \hline 0 & \mathbf{A}' \end{array}\right]. \tag{4.17}$$

The spatial components $\mathbf{A}'$ are given by (4.3) and (4.1b) with all quantities dashed. The time component $\bar{A}$ is arbitrary because of (2.10). The time component $\lambda = 0$ of (4.15) in the comoving frame is the identity $0 = 0$; the spatial components are (4.9) with all quantities dashed. The dispersion equation (4.16) becomes $\boldsymbol{n}' \cdot \hat{\boldsymbol{\varepsilon}}' \cdot \mathbf{A}' \cdot \boldsymbol{n}' = 0$ (4.10). Analogous considerations lead to the transition of (4.13), (4.14) to the (dashed) expressions (4.4), (4.6) in the comoving frame.

The considerations following the dispersion equation (4.10) regarding the elimination of $\boldsymbol{E}$ instead of $\boldsymbol{H}$ can be generalized to moving media. Three-dimensional vectors and tensors have to be replaced by their four-dimensional generalizations.

5. Representation of Polarization Relations and Dispersion Equations with Eigenvalues and Eigenvectors

For magnetically isotropic media with $\boldsymbol{\mu} = \mu \mathbf{I}$, i.e. $\hat{\boldsymbol{\varkappa}} = \mathbf{I}$ (3.2), the three-dimensional tensors $\mathbf{C}$ (4.1b) and $\mathbf{A}$ (4.3) simplify considerably and we denote them by

$$\hat{\mathbf{C}} := n^2 \mathbf{I} - \hat{\boldsymbol{\varepsilon}}, \tag{5.1}$$

$$\hat{\mathbf{A}} := \hat{\mathbf{C}}^{-1} \det \hat{\mathbf{C}} = (n^2 \mathbf{I} - \hat{\boldsymbol{\varepsilon}})^{-1} \det \hat{\mathbf{C}}. \tag{5.2}$$

Since the tensors $\hat{\boldsymbol{\varepsilon}}$ and $\hat{\mathbf{C}}$ differ only by a tensor proportional to the unit tensor $\mathbf{I}$, they both have the same left and right eigenvectors, say $\boldsymbol{y}^{a'}$ and $\boldsymbol{x}_{a'}$ $(a' = 1, 2, 3)$, respectively. They are also the eigenvectors of $\hat{\mathbf{A}}$ (5.2). They satisfy the biorthonormality relations

$$\boldsymbol{y}^{a'} \cdot \boldsymbol{x}_{b'} = \delta^{a'}_{b'} \tag{5.3a}$$

and the completeness relation

$$\sum_{a'=1}^{3} \boldsymbol{x}_{a'} \boldsymbol{y}^{a'} = \mathbf{I} \tag{5.3b}$$

for the three projectors $\boldsymbol{x}_{a'} \boldsymbol{y}^{a'}$. With this relation, with the diagonal representations

$$\hat{\boldsymbol{\varepsilon}} = \sum_{a'=1}^{3} \hat{\varepsilon}_{a'} \boldsymbol{x}_{a'} \boldsymbol{y}^{a'}, \quad \hat{\mathbf{C}} = \sum_{a'=1}^{3} \hat{C}_{a'} \boldsymbol{x}_{a'} \boldsymbol{y}^{a'}, \tag{5.4a}$$

$$\hat{\mathbf{A}} = \sum_{a'=1}^{3} \hat{A}_{a'} \boldsymbol{x}_{a'} \boldsymbol{y}^{a'}, \tag{5.4b}$$

where $\hat{\varepsilon}_{a'}$, $\hat{C}_{a'}$ and $\hat{A}_{a'}$ are the eigenvalues of $\hat{\boldsymbol{\varepsilon}}$, $\hat{\mathbf{C}}$ and $\hat{\mathbf{A}}$, respectively, the definitions (5.1), (5.2) for the tensors $\hat{\mathbf{C}}$ and $\hat{\mathbf{A}}$ can be written as relations between the corresponding eigenvalues:

$$\hat{C}_{a'} = n^2 - \hat{\varepsilon}_{a'}, \tag{5.5}$$

$$\hat{A}_{a'} = \det \hat{\mathbf{C}} / (n^2 - \hat{\varepsilon}_{a'}) = (n^2 - \hat{\varepsilon}_{b'})(n^2 - \hat{\varepsilon}_{c'})$$

with $\quad a' \neq b' \neq c'$. $\quad$ (5.6)

The polarization relation (4.4) and the dispersion equations (4.6) and (4.10) become ([6], eqs. 6.12 and 6.13)

$$\boldsymbol{E} = \sum_{a'} \boldsymbol{x}_{a'} \frac{\boldsymbol{y}^{a'} \cdot \boldsymbol{n}}{n^2 - \hat{\varepsilon}_{a'}} \boldsymbol{n} \cdot \boldsymbol{E} \tag{5.7}$$

$$\left(\sum_{a'} \frac{\boldsymbol{n} \cdot \boldsymbol{x}_{a'} \boldsymbol{y}^{a'} \cdot \boldsymbol{n}}{n^2 - \hat{\varepsilon}_{a'}} - 1 \right) \det \hat{\mathbf{C}} = 0, \tag{5.8}$$

$$0 = \sum_{a'} \hat{\varepsilon}_{a'} \hat{A}_{a'} \boldsymbol{n} \cdot \boldsymbol{x}_{a'} \boldsymbol{y}^{a'} \cdot \boldsymbol{n} \tag{5.9}$$

$$= \sum_{a'} \frac{\boldsymbol{n} \cdot \boldsymbol{x}_{a'} \boldsymbol{y}^{a'} \cdot \boldsymbol{n}}{\dfrac{1}{\hat{\varepsilon}_{a'}} - \dfrac{1}{n^2}} \frac{\det \hat{\mathbf{C}}}{n^2}$$

$$= \sum_{a' \neq b' \neq c'} \hat{\varepsilon}_{a'} (n^2 - \hat{\varepsilon}_{b'})(n^2 - \hat{\varepsilon}_{c'}) \boldsymbol{n} \cdot \boldsymbol{x}_{a'} \boldsymbol{y}^{a'} \cdot \boldsymbol{n}.$$

For symmetric effective dielectric tensors $\hat{\boldsymbol{\varepsilon}}$ (like those for crystals) left and right eigenvectors (for the same eigenvalue) are equal $(\boldsymbol{x}_{a'} = \boldsymbol{y}^{a'})$ and therefore define principal axes. The expression $\boldsymbol{x}_{a'} \cdot \boldsymbol{n}/n$ is the cosine of the angle between the principle axis $\boldsymbol{x}_{a'}$ and the wave normal $\boldsymbol{n}/n = \boldsymbol{k}/k$. In this case the dispersion equation (5.9) is Fresnel's equation of wave normals ([7], § 14.2, Equation 21).

The generalization of the preceding calculations to four dimensions is straightforward. The definitions (cf. (4.11))

$$\hat{C}^{\nu}{}_{\lambda} := n_{\gamma} n^{\gamma} \delta^{\nu}_{\lambda} - \hat{\varepsilon}^{\nu}{}_{\lambda}, \tag{5.10}$$

$$\hat{A}^{\mu}{}_{\nu} \hat{C}^{\nu}{}_{\lambda} = \delta^{\mu}_{\lambda} \det \hat{C} \tag{5.11}$$

can be written with the diagonal representations

$$\hat{\varepsilon}^{\nu}{}_{\lambda} = \sum_{\alpha'=0}^{3} \hat{\varepsilon}_{\alpha'} x^{\nu}_{\alpha'} y^{\alpha'}_{\lambda}, \quad \hat{C}^{\varkappa}{}_{\lambda} = \sum_{\alpha'=0}^{3} \hat{C}_{\alpha'} x^{\varkappa}_{\alpha'} y^{\alpha'}_{\lambda} \tag{5.12a}$$

$$\hat{A}^{\varkappa}{}_{\lambda} = \sum_{\alpha'=0}^{3} \hat{A}_{\alpha'} x^{\varkappa}_{\alpha'} y^{\alpha'}_{\lambda} \tag{5.12b}$$

as relations between eigenvalues:

$$\hat{C}_{\alpha'} = n_{\gamma} n^{\gamma} - \hat{\varepsilon}_{\alpha'}, \tag{5.13}$$

$$\hat{A}_{\alpha'} = \det \hat{C}/(n_\gamma n^\gamma - \hat{\varepsilon}_{\alpha'}) \,. \tag{5.14}$$

The left and right eigenvectors $y^{\alpha'}_\lambda$ and $x^\nu_{\alpha'}$ ($\alpha' = 0, 1, 2, 3$) are again biorthonormal and their projectors $x^\nu_{\alpha'}\, y^{\alpha'}_\lambda$ form a complete set.

The representation (2.10) of the effective dielectric tensor $\hat{\varepsilon}'^\nu{}_\lambda$ in the comoving frame leads to the following representation of the eigenvectors in this frame:

$$x'^\nu_0 = u'^\nu \,, \qquad y'^0_\mu = u'_\mu \,, \tag{5.15a}$$

$$x'^\nu_{a'} = [0 \,\vdots\, \boldsymbol{x}'_{a'}] \,, \quad y'^{\alpha'}_\mu = [0 \,\vdots\, \boldsymbol{y}'^{a'}] \,, \quad a' = 1, 2, 3 \tag{5.15b}$$

with $\boldsymbol{x}'_{a'}$ and $\boldsymbol{y}'^{a'}$ as the eigenvectors of the three-dimensional effective dielectric tensor expressed in the comoving frame. The eigenvalue $\hat{\varepsilon}_0{}'$ and thus $\hat{C}_0{}'$ is arbitrary (2.10). The components of the eigenvectors in the observer's frame are obtained by Lorentz transformation from the components in the comoving frame.

With the diagonal representations (5.12) and the completeness relation for the projectors $x^\nu_{\alpha'}\, y^{\alpha'}_\lambda$ the polarization relations (4.13) and the dispersion equations (4.14) and (4.16) become

$$E^\mu = \sum_{\alpha'} x^\mu_{\alpha'} \frac{y^{\alpha'}_\nu n^\nu}{n_\gamma n^\gamma - \hat{\varepsilon}_{\alpha'}} n_\lambda E^\lambda \tag{5.16}$$

$$\left(\sum_{\alpha'} \frac{n_\mu x^\mu_{\alpha'} y^{\alpha'}_\nu n^\nu}{n_\gamma n^\gamma - \hat{\varepsilon}_{\alpha'}} - 1 \right) \det \hat{C} = 0 \,, \tag{5.17}$$

$$\begin{aligned} 0 &= \sum_{a'} \hat{\varepsilon}_{\alpha'} \hat{A}_{\alpha'} n_\mu x^\mu_{\alpha'} y^{\alpha'}_\nu n^\nu \\ &= \sum_{a'} \frac{n_\mu x^\mu_{\alpha'} y^{\alpha'}_\nu n^\nu}{\dfrac{1}{\hat{\varepsilon}_{\alpha'}} - \dfrac{1}{n_\gamma n^\gamma}} \frac{\det \hat{C}}{n_\gamma n^\gamma} \\ &= (n_\gamma n^\gamma - \hat{\varepsilon}_0) \sum_{a' \neq b' \neq c'} \hat{\varepsilon}_{a'} (n_\gamma n^\gamma - \hat{\varepsilon}_{b'}) \\ &\quad \cdot (n_\gamma n^\gamma - \hat{\varepsilon}_{c'}) \, n_\mu x^\mu_{\alpha'} y^{\alpha'}_\nu n^\nu \,. \end{aligned} \tag{5.18}$$

Because of $n^\nu u_\nu = 0$ (3.10a) the vector n^ν is orthogonal to the eigenvector $x^\nu_0 = u^\nu = y^\nu_0$ (5.15a). Therefore the above sums of four terms ($\alpha' = 0, 1, 2, 3$) reduce to sums of three terms ($a' = 1, 2, 3$). Hence the arbitrary eigenvalue $\hat{\varepsilon}_0$ does not contribute to polarization and dispersion, and the last expression of the dispersion equation (5.18) is justified.

In the comoving frame Eqs. (3.9b) and (5.15) yield

$$\frac{n'_\mu x'^\mu_{a'}}{n'^\nu_\gamma n'^\gamma} = \frac{\boldsymbol{n}' \cdot \boldsymbol{x}'_{a'}}{n'^2} \,, \quad \frac{y'^{a'}_\nu n'^\nu}{n'_\gamma n'^\gamma} = \frac{\boldsymbol{y}'^{a'} \cdot \boldsymbol{n}'}{n'^2} \tag{5.19}$$

and the Eqs. (5.16), (5.17), (5.18) become (5.7), (5.8), (5.9) with all quantities dashed.

For symmetric effective dielectric tensors $\hat{\varepsilon}^\varkappa{}_\mu = \hat{\varepsilon}_\mu{}^\varkappa$ left and right eigenvectors are equal, hence $n_\mu x^\mu_{a'} = y^{a'}_\nu n^\nu$. In the comoving frame (5.18) becomes Fresnel's wave normal equation.

6. Principal Modes and Their Principal Polarizations

In crystal optics principal waves, i.e. waves propagating parallel to one of the principal dielectric axes (eigenvectors), say $\boldsymbol{x}_{a'}$, have particular properties. Their two values of n^2 are equal to the eigenvalues $\varepsilon_{b'}/\varepsilon_0$ and $\varepsilon_{c'}/\varepsilon_0$ ($a' \neq b' \neq c'$) of the relative dielectric tensor $\boldsymbol{\epsilon}/\varepsilon_0$ ([7], Sect. 14.2.3a). To obtain this result the symmetry of the dielectric tensor $\boldsymbol{\epsilon}$ is essential, i.e. the equality of the left eigenvectors $\boldsymbol{y}^{a'}$ with the right eigenvectors $\boldsymbol{x}_{a'}$. To generalize this result to media with non-symmetric effective dielectric tensors (like magnetized plasmas), the definition of principal waves must be reconsidered.

We investigate the condition under which one of the two solutions of the dispersion equation (5.9) for n^2 equals an eigenvalue of $\hat{\boldsymbol{\epsilon}}$, say $\hat{\varepsilon}_{a'}$. To do this we write this dispersion equation as

$$\begin{aligned} 0 &= \hat{\varepsilon}_{a'} (n^2 - \hat{\varepsilon}_{b'})(n^2 - \hat{\varepsilon}_{c'}) \, \boldsymbol{n} \cdot \boldsymbol{x}_{a'} \boldsymbol{y}^{a'} \cdot \boldsymbol{n} \\ &\quad + (n^2 - \hat{\varepsilon}_{a'}) [\hat{\varepsilon}_{b'} (n^2 - \hat{\varepsilon}_{c'}) \, \boldsymbol{n} \cdot \boldsymbol{x}_{b'} \boldsymbol{y}^{b'} \cdot \boldsymbol{n} \\ &\quad + \hat{\varepsilon}_{c'} (n^2 - \hat{\varepsilon}_{b'}) \, \boldsymbol{n} \cdot \boldsymbol{x}_{c'} \boldsymbol{y}^{c'} \cdot \boldsymbol{n}] \,. \end{aligned} \tag{6.1}$$

We obtain the desired solution if and only if we put $\boldsymbol{n} \cdot \boldsymbol{x}_{a'} = 0$ or $\boldsymbol{n} \cdot \boldsymbol{y}^{a'} = 0$, i.e. for a wave propagating perpendicular to either $\boldsymbol{x}_{a'}$ or $\boldsymbol{y}^{a'}$. In the limit $n^2 \to \hat{\varepsilon}_{a'}$ the difference $n^2 - \hat{\varepsilon}_{a'}$ is proportional to $\boldsymbol{n} \cdot \boldsymbol{x}_{a'}$, or $\boldsymbol{n} \cdot \boldsymbol{y}^{a'}$, respectively. In the first case ($\boldsymbol{n} \cdot \boldsymbol{x}_{a'} \to 0$) we conclude from the dispersion equation (6.1) that the ratio $(n^2 - \hat{\varepsilon}_{a'})/\boldsymbol{n} \cdot \boldsymbol{y}^{a'}$ is proportional to the vanishing of $\boldsymbol{n} \cdot \boldsymbol{x}_{a'}$ in the limit $n^2 \to \hat{\varepsilon}_{a'}$. With this result the polarization relation (5.7) requires the vanishing of $\boldsymbol{n} \cdot \boldsymbol{E}$, i.e. the transversality of $\boldsymbol{E}$, and yields as polarization

$$\boldsymbol{E} \sim \boldsymbol{x}_{a'} \,. \tag{6.2}$$

In the second case ($\boldsymbol{n} \cdot \boldsymbol{y}^{a'} \to 0$) the ratio

$$(n^2 - \hat{\varepsilon}_{a'})/\boldsymbol{n} \cdot \boldsymbol{y}^{a'}$$

is in general finite. Therefore the polarization relation (5.7) does in general not require the vanishing of $\boldsymbol{n} \cdot \boldsymbol{E}$ and does not yield a simple polarization expression like (6.2). Therefore we define a "principal mode" as a wave with $n^2 = \hat{\varepsilon}_{a'}$ propagating perpendicular to a right eigenvector $\boldsymbol{x}_{a'}$. Its "principal polarization" is then $\boldsymbol{E} \sim \boldsymbol{x}_{a'}$ (6.2).

In the special case of uniaxial media, i.e. uniaxial crystals and gyrotropic media (e.g. magnetized plasmas) the symmetry axis plays a particular role. Therefore one pair of eigenvectors, say $\boldsymbol{x}_{c'}$ and $\boldsymbol{y}^{c'}$, is parallel to it, the other eigenvectors are all perpendicular to it. For waves propagating parallel to $\boldsymbol{x}_{c'} = \boldsymbol{y}^{c'}$ the dispersion equation (6.1) becomes

$$0 = (n^2 - \hat{\varepsilon}_{a'})\,\hat{\varepsilon}_{c'}\,(n^2 - \hat{\varepsilon}_{b'})\,n^2 \tag{6.3}$$

with the two solutions

$$n^2 = \hat{\varepsilon}_{a'} \quad \text{and} \quad n^2 = \hat{\varepsilon}_{b'}\,. \tag{6.4}$$

The polarization relation (5.7) requires again the transversality of $\boldsymbol{E}$ and yields the two principal polarizations ([7], Sect. 14.2.3a)

$$\boldsymbol{E} \sim \boldsymbol{x}_{a'} \quad \text{and} \quad \boldsymbol{E} \sim \boldsymbol{x}_{b'} \tag{6.5}$$

for principal modes propagating parallel to $\boldsymbol{x}_{c'} = \boldsymbol{y}^{c'}$ obeying (6.4).

In crystals with the identical cartesian sets of eigenvectors $\boldsymbol{x}_{a'} = \boldsymbol{y}^{a'}$ ($a' = 1, 2, 3$) a principal mode propagates always perpendicular to two pairs of eigenvectors, say $\boldsymbol{x}_{a'} = \boldsymbol{y}^{a'}$ and $\boldsymbol{x}_{b'} = \boldsymbol{y}^{b'}$, hence parallel to the third pair, say $\boldsymbol{x}_{c'} = \boldsymbol{y}^{c'}$. Then (6.4), (6.5) hold for all three principal axes $c' = 1, 2, 3$.

The generalization of these results to moving media is formally simple. The scalar products $\boldsymbol{n} \cdot \boldsymbol{x}$ and $\boldsymbol{y} \cdot \boldsymbol{n}$ in (6.1) are replaced by $n_\nu x^\nu$ and $y_\nu n^\nu$. A principal mode has a four-vector n_ν perpendicular to a right eigenvector, say $x^\nu_{a'}$, and has the solution

$$n_\gamma n^\gamma = \hat{\varepsilon}_{a'} \quad \text{for} \quad n_\nu x^\nu_{a'} = 0 \tag{6.6}$$

of the dispersion equation (5.18). Since n_ν is always perpendicular to u^ν (3.10a) the direction of n_ν is determined by (3.10a) and (6.6). The polarization relation (5.16) requires the vanishing of $n_\lambda E^\lambda$ and yields the principal polarization

$$E^\mu \sim x^\mu_{a'} \tag{6.7}$$

for the principal mode obeying (6.6). Since E^λ is perpendicular to u_μ (2.15a) the vanishing of $k_\lambda E^\lambda$ can be deduced from the definition (3.9a) of n^ν and the vanishing of $n_\lambda E^\lambda$. But the vanishing of $k_\lambda E^\lambda$ does in general not imply the vanishing of $\boldsymbol{k} \cdot \boldsymbol{E}$, as can be seen from the definition (2.15a) for E^λ.

7. General Calculation of the Polarization Relations

To calculate the explicit representation of the polarization relation (4.4) we rewrite (4.4) as

$$\boldsymbol{E} = \mathbf{A} \cdot \boldsymbol{n}(n\,E_L/\det \mathbf{C}) \quad \text{with}$$
$$E_L := \boldsymbol{E} \cdot \boldsymbol{n}/n \tag{7.1}$$

and introduce a transverse base $\boldsymbol{g}_{A'}$, $\boldsymbol{g}^{B'}$ ($A' = 1, 2$; $B' = 1, 2$) with

$$\boldsymbol{g}_{A'} \cdot \boldsymbol{g}^{B'} = \delta^{B'}_{A'}\,, \quad \boldsymbol{g}_{A'} \cdot \boldsymbol{n} = 0 = \boldsymbol{g}^{B'} \cdot \boldsymbol{n}\,. \tag{7.2}$$

Subsequent multiplication of Eq. (7.1) with the transverse vector $\boldsymbol{g}_1$ and $\boldsymbol{g}_2$, respectively, leads with $E_{A'} := \boldsymbol{g}_{A'} \cdot \boldsymbol{E}$ to the polarization relations in the form

$$E_1 : E_2 : E_L = \boldsymbol{g}_1 \cdot \mathbf{A} \cdot \boldsymbol{n} : \boldsymbol{g}_2 \cdot \mathbf{A} \cdot \boldsymbol{n} : (1/n) \det \mathbf{C} \tag{7.3}$$

To generalize these polarization relations to moving media we introduce two space-like four-dimensional vectors

$$g^\mu_{A'} := [0 \,\vdots\, \boldsymbol{g}_{A'}]\,, \quad g^{B'}_{\mu'} := [0 \,\vdots\, \boldsymbol{g}^{B'}]\,, \tag{7.4}$$

whose space-parts $\boldsymbol{g}_{A'}$, $\boldsymbol{g}^{B'}$ form a transverse base, cf. (7.2).

Subsequent contraction of the polarization relation (4.13) with g^μ_1 and g^μ_2, respectively, yields

$$g^\mu_{A'}\,E_\mu = g^\mu_{A'}\,A_\mu{}^\nu\,n_\nu(n_\lambda\,E^\lambda/\det C)\,. \tag{7.5}$$

But the four-field

$$E^\lambda = \gamma\left[\frac{\boldsymbol{v}}{c} \cdot \boldsymbol{E} \,\vdots\, \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right] \tag{2.15a}$$

contains the magnetic flux $\boldsymbol{B}$ in addition to the electric field $\boldsymbol{E}$. We eliminate $\boldsymbol{B}$ with the three-dimensional Maxwell equation $\boldsymbol{k} \times \boldsymbol{E} = \omega \boldsymbol{B}$ (2.1b) and obtain

$$E^\lambda = \gamma\left[\frac{\boldsymbol{v}}{c} \cdot \boldsymbol{E} \,\vdots\, \frac{\omega - \boldsymbol{v} \cdot \boldsymbol{k}}{\omega}\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \cdot \boldsymbol{E}\frac{c\,\boldsymbol{k}}{\omega}\right]. \tag{7.6}$$

Thus the inner products

$$g^\mu_{A'}E_\mu = \frac{\gamma}{\omega}(\omega - \boldsymbol{v} \cdot \boldsymbol{k})\boldsymbol{g}_{A'} \cdot \boldsymbol{E} = -\frac{c}{\omega}k_\alpha\,u^\alpha\,E_{A'} \tag{7.7}$$

are proportional to the transverse components $E_{A'} := \boldsymbol{g}_{A'} \cdot \boldsymbol{E}$ of the electric field. Therefore the transverse polarization relations can be obtained from (7.5) in covariant form as

$$E_1 : E_2 = g^\mu_1\,A_\mu{}^\nu\,n_\nu : g^\mu_2\,A_\mu{}^\nu\,n_\nu\,. \tag{7.8}$$

To include the longitudinal component $E_L := \boldsymbol{E} \cdot \boldsymbol{k}/|\boldsymbol{k}|$ (7.1) we have to show how it is contained in the factor $n_\lambda E^\lambda$ of the polarization relation (7.5). With (3.11), (7.6), (7.7) we can write

$$n_\lambda E^\lambda = \frac{k_\lambda E^\lambda}{-k_\varkappa u^\varkappa} = \frac{1}{-k_\varkappa u^\varkappa} \frac{c\gamma}{\omega} \cdot \left[\left(k^2 - \frac{\omega^2}{c^2}\right)\frac{\boldsymbol{v}}{c} \cdot \boldsymbol{E} + \frac{\omega - \boldsymbol{v}\cdot\boldsymbol{k}}{c} |\boldsymbol{k}| E_L\right]. \tag{7.9}$$

Next we decompose the scalar product $\boldsymbol{v}\cdot\boldsymbol{E}$ in transverse and longitudinal parts as

$$\gamma \frac{\boldsymbol{v}}{c} \cdot \boldsymbol{E} = g^1_\beta u^\beta E_1 + g^2_\beta u^\beta E_2 + \gamma \frac{\boldsymbol{v}}{c} \cdot \frac{\boldsymbol{k}}{|\boldsymbol{k}|} E_L , \tag{7.10}$$

insert this in (7.9) and obtain (with $k^2 - \omega^2/c^2 = k_\alpha k^\alpha$)

$$(-k_\varkappa u^\varkappa) n_\lambda E^\lambda = k_\alpha k^\alpha \frac{c}{\omega} u^\beta (g^1_\beta E_1 + g^2_\beta E_2) + \gamma \frac{\boldsymbol{k}}{|\boldsymbol{k}|}\left(\boldsymbol{k} - \frac{\omega \boldsymbol{v}}{c^2}\right) E_L . \tag{7.11}$$

If we express the left-hand side of the polarization relation (7.5) by (7.7) and the right-hand side by (7.11) we obtain a system of two homogeneous linear equations for the three components E_1, E_2, E_L. The relations $E_1 : E_2 : E_L$ are then given by the ratios of the three corresponding 2×2 matrices calculated from the six coefficients of this linear system. After some rearrangements we obtain as polarization relations for all three components

$$E_1 : E_2 : E_L = g^\mu_1 A_\mu{}^\nu n_\nu : g^\mu_2 A_\mu{}^\nu n_\nu : \frac{c}{\omega} |\boldsymbol{k}| \cdot \frac{(k_\alpha u^\alpha)^2 \det C - k^\alpha k_\alpha u^\beta (g^1_\beta g^\mu_1 + g^2_\beta g^\mu_2) A_\mu{}^\nu n_\nu}{\gamma \boldsymbol{k} \cdot \left(\boldsymbol{k} - \frac{\omega \boldsymbol{v}}{c^2}\right)} . \tag{7.12}$$

This is the generalization of (7.3) for moving media. The longitudinal part cannot be expressed in a completely covariant form.

8. Calculation of the Polarization Relations by Transformation of the Field Components

In the preceding section polarization relations for moving media have been derived. The resulting expressions (7.12) are rather involved and make it difficult to recognize explicitly the influence of the convection on the polarization. However, if we consider only modes already present in the comoving frame we can calculate the polarizations in the observer's frame by Lorentz transforming the field components in the comoving frame.

The transformation of a three-dimensional field vector (e.g. the electric field $\boldsymbol{E}$) is given by ([8], Eqs. (2.16b), (2.6a), (2.6h), (2.6j))

$$\boldsymbol{E} = \gamma \mathbf{V} \cdot (\boldsymbol{E}' - \boldsymbol{v} \times \boldsymbol{B}') \tag{8.1a}$$

with

$$\mathbf{V} := \mathbf{I} - \frac{\boldsymbol{v}\boldsymbol{v}}{v^2} + \frac{1}{\gamma}\frac{\boldsymbol{v}\boldsymbol{v}}{v^2} . \tag{8.1b}$$

Eliminating the magnetic flux $\boldsymbol{B}'$ by Maxwell's equation (2.1b) one obtains the transformation of the electric field alone ([9], p. 229, Eq. (22))

$$\boldsymbol{E} = \mathbf{A} \cdot \boldsymbol{E}' . \tag{8.2a}$$

The transformation matrix $\mathbf{A}$ is given by

$$\mathbf{A} := \gamma \frac{\omega' + \boldsymbol{k}' \cdot \boldsymbol{v}}{\omega'} \mathbf{I} - (\gamma - 1)\frac{\boldsymbol{v}\boldsymbol{v}}{v^2} - \gamma \frac{\boldsymbol{k}'}{\omega'} \boldsymbol{v} . \tag{8.2b}$$

We substitute $\boldsymbol{k}'$ and ω' by their Lorentz transformed counterparts ([8], Eqs. (3.26), (2.2), (2.6e))

$$\boldsymbol{k}' = \mathbf{V}^{-1} \cdot \left(\boldsymbol{k} - \frac{\omega \boldsymbol{v}}{c^2}\right), \quad \omega' = \gamma(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) , \tag{8.3}$$

and obtain after some manipulations

$$\mathbf{A} = \left[\omega \mathbf{I} - \gamma \boldsymbol{k}\boldsymbol{v} - \omega(\gamma - 1)\frac{\boldsymbol{v}\boldsymbol{v}}{v^2}\right] \Big/ \gamma(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) . \tag{8.4}$$

Choosing a coordinate system with the $\boldsymbol{e}_3$-axis in the direction of the three-dimensional wave vector $\boldsymbol{k} = (0, 0, k)$, we can define transverse and longitudinal polarizations Q_T and Q_L as

$$Q_T := E_1/E_2 , \quad Q_L := E_3/E_2 . \tag{8.5}$$

In order to simplify the expressions we fix the transverse basis system so that $\boldsymbol{e}_2$ is perpendicular to the velocity $\boldsymbol{v} = (v_T, 0, v_L)$. Polarizations with respect to another transverse coordinate system can be easily calculated by an appropriate transformation. According to the special choice of the coordinate system the following components of the transformation matrix (8.4) vanish:

$$A_{21} = A_{12} = A_{23} = A_{32} = 0 . \tag{8.6}$$

Thus the polarizations Q_T and Q_L in the observer's frame can be expressed by the corresponding polarizations in the comoving frame as follows:

$$Q_T := \frac{E_1}{E_2} = \frac{A_{11}E_1' + A_{13}E_3'}{A_{22}E_2'} = \frac{A_{11}}{A_{22}}Q_T' + \frac{A_{13}}{A_{22}}Q_L' \tag{8.7a}$$

$$Q_L := \frac{E_3}{E_2} = \frac{A_{31}E_1' + A_{33}E_3'}{A_{22}E_2} = \frac{A_{31}}{A_{22}}Q_T' + \frac{A_{33}}{A_{22}}Q_L' . \tag{8.7b}$$

The above expressions show that one has to know both polarizations in the comoving frame to calculate one of the polarizations in the observer's frame by transformation. Insertion of the explicit expressions for the components of A (8.4) into Eqs. (8.7) leads to

$$Q_T = \left[1 - \frac{v_T^2}{v^2}(1-\gamma)\right]Q_T' - (1-\gamma)\frac{v_T v_L}{v^2}Q_L' , \tag{8.8a}$$

$$Q_L = \left[1 - \frac{v_L^2}{v^2}(1-\gamma) - \gamma\frac{\boldsymbol{k}\cdot\boldsymbol{v}}{\omega}\right]Q_L' - \left[(1-\gamma)\frac{v_T v_L}{v^2} + \gamma\frac{|\boldsymbol{k}\times\boldsymbol{v}|}{\omega}\right]Q_T' . \tag{8.8b}$$

If we neglect terms of order v^2/c^2 we obtain

$$Q_T = Q_T' + O(v^2/c^2) , \tag{8.9a}$$

$$Q_L = \frac{\omega - \boldsymbol{k}\cdot\boldsymbol{v}}{\omega}Q_L' - \frac{|\boldsymbol{k}\times\boldsymbol{v}|}{\omega}Q_T' + O\left(\frac{v^2}{c^2}\right). \tag{8.9b}$$

Thus the transverse polarization Q_T is influenced by the convection only through terms of order v^2/c^2. The coeffcients in Eq. (8.8a) depend only on the velocity and not on $\boldsymbol{k}/\omega$. They are real. Thus linear polarizations Q_T', Q_L' in the comoving frame yield a linear transverse polarization in the observer's frame.

The longitudinal polarization is much more influenced by the convection because of the terms $\boldsymbol{v}\cdot\boldsymbol{k}/\omega$, $|\boldsymbol{v}\times\boldsymbol{k}|/\omega$ in (8.9b). These terms, proportional to the absolute value of the wave vector, can take large and in general complex values (for media with losses). The transformation can alter the polarization decisively, i.e. a linear polarization in the comoving frame becomes elliptical.

In the special case of the wave propagation parallel to the velocity $\boldsymbol{v}$ ($v_T = 0$) the exact Eqs. (8.8) become

$$Q_T = Q_T' , \quad Q_L = \gamma\frac{\omega - k v}{\omega}Q_L' \quad \text{for} \quad \boldsymbol{k}\parallel\boldsymbol{v} . \tag{8.10}$$

The transverse polarization remains unaltered. The longitudinal polarization is proportional to its value in the comoving frame. The proportionality factor is the relativistic Doppler ratio. For lossy media it becomes complex. Then a linear or circular polarization in the comoving frame becomes elliptical in the observer's frame.

In the case of the wave propagation perpendicular to the convection ($v_L = 0$) Eqs. (8.8) become

$$Q_T = \gamma Q_T' , \quad Q_L = -\gamma\frac{k v}{\omega}Q_T' + Q_L' \quad \text{for } \boldsymbol{k}\perp\boldsymbol{v} . \tag{8.11}$$

This shows again the strong influence of the convection on the longtudinal polarization. The transverse polarizations Q_T, Q_T' are related with γ as proportionality factor.

9. Concluding Remarks

In this paper the main aim had been to derive results for media whose anisotropy (in the comoving frame) is as general as possible. Therefore Sects. 2, 3, 4, 7, 8 hold for media anisotropic for electric as well as for magnetic properties. In Sects. 5 and 6 eigenvectors and eigenvalues have been introduced restricting the results therein to media magnetically isotropic (in the comoving frame). By a suitable change of variables the results of Sects. 5 and 6 can as well be used for media electrically isotropic but magnetically anisotropic (in the comoving frame). No assumptions have been made about the (left and right) eigenvectors of the material tensors. In a subsequent paper the results will be specialized for gyrotropic media (without spatial dispersion), e.g. magnetized cold plasmas.

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[1] H. Hebenstreit, Calculation of Covariant Dispersion Equations for Moving Plasmas. Z. Naturforsch. **34a**, 155 (1979b).
[2] J. D. Jackson, Classical Electrodynamics (2nd ed.), John Wiley, New York 1975.
[3] E. Schmutzer, Relativistische Physik, Akad. Verlagsgesellschaft, Leipzig 1968.
[4] H. Hebenstreit, Constitutive Relations for Moving Plasmas. Z. Naturforsch. **34a**, 147 (1979a).
[5] J. W. Gibbs, Vector Analysis, Dover, New York 1960.
[6] K. Rawer, K. Suchy, Radio-Observations of the Ionosphere, Handbuch der Physik 49/2, 1—546 (ed. S. Flügge), Springer-Verlag, Heidelberg 1967.
[7] M. Born and E. Wolf, Principles of Optics, Pergamon, London 1970.
[8] J. A. Kong, Theory of Electromagnetic Waves, John Wiley, New York 1975.
[9] S. Tamor, The Absorption Coefficients of a Magnetized Plasma for Cyclotron Radiation, Nuclear Fusion **18**, 229 (1978).